## fn discrete\_laplacian\_scale\_to\_accuracy

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This document contains materials associated with discrete\_laplacian\_scale\_to\_accuracy.

**Definition 0.1.** Let z be the true value of the statistic and X be the random variable the noisy release is drawn from. Define Y = |X - z|, the distribution of DP errors. Then for any statistical significance level alpha, denoted  $\alpha \in [0, 1]$ , and accuracy, denoted  $a \ge 0$ ,

$$\alpha = P[Y \ge a] \tag{1}$$

**Theorem 0.2.** For any scale  $\geq 0$  denoted s, when  $X \sim \mathcal{L}_{\mathbb{Z}}(z,s)$ ,

$$a = s \cdot \ln(2/(\alpha(e^{1/s} + 1))) + 1 \tag{2}$$

*Proof.* Consider that the distribution of  $(X - z) \sim \mathcal{L}_{\mathbb{Z}}(0, s)$ . Then the PMF of Y is:

$$\forall y \ge 0 \qquad g(y) = (1 + 1[y \ne 0]) \frac{1 - e^{-1/s}}{1 + e^{-1/s}} e^{-y/s} \tag{3}$$

The purpose of the indicator function is to avoid double-counting zero. Now derive an expression for  $\alpha$ :

$$\begin{aligned} \alpha &= P[Y \ge a] \\ &= 1 - P[Y < a] \\ &= 1 - \sum_{y=0}^{a-1} g(y) \\ &= 1 - \frac{1 - e^{-1/s}}{1 + e^{-1/s}} \left( 1 + 2 \sum_{y=0}^{a-1} e^{-y/s} \right) \\ &= 2 \frac{e^{(1-a)/s}}{e^{1/s} + 1} \end{aligned}$$

where g(y) is the distribution of Y

Invert to solve for a:

$$2\frac{e^{(1-a)/s}}{e^{1/s}+1} = \alpha$$
$$e^{(1-a)/s} = \alpha(e^{1/s}+1)/2$$
$$a = 1 - s \cdot \ln(\alpha(e^{1/s}+1)/2)$$
$$a = s \cdot \ln(2/(\alpha(e^{1/s}+1))) + 1$$